A GEOMETRIC EXAMPLE OF AN INDETERMINATE FORM.

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In the prolate ellipsoid of revolution generated by turning the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ around the x-axis be cut by the plane $x=c$ through one focus, the area of the smaller part, between this focal plane and the nearer vertex, is given by the integral

$$S=\frac{2\pi b}{a^2}\int_c^a \sqrt{[a^4-(a^2-b^2)x^2]}\,dx.$$

When the constants are expressed in terms of the eccentricity e and the distance q between the focus and vertex, by means of the relations

$$c=ae,\ b=a\sqrt{[1-e^2]},\ q=a(1-e),$$

the integral reduces to

$$(1)\qquad S=\pi q^2\sqrt{\frac{1+e}{(1-e)^3}}\{\sqrt{[1-e^2]}-e\sqrt{[1-e^4]}+\frac{\sin^{-1}e-\sin^{-1}e^2}{e}\}.$$

If now the distance q be kept constant, the limiting surface (for $e=1$) is the paraboloid of revolution generated by $y^2=4qx$ (with a convenient change of axes); and the corresponding area is

$$S_1=\frac{\pi}{q}\int_0^{2q} y\sqrt{[y^2+4q^2]}\,dy$$

which reduces to

$$(2)\qquad S_1=\frac{8\pi q^2}{3}(\sqrt{8}-1).$$

It is thus apparent that as e approaches unity the limit S should be S_1. But the expression for S is then formally indeterminate, and verification is not directly practicable by the customary method of successive differentiation with respect to e, the reason being the presence of the factor $(1-e)$ raised to a fractional power, which cannot be reduced to a constant by repeated differentiation. It may be noticed also that the point $e=1$ is a branch-point for $\sin^{-1}e$ and for $\sin^{-1}e^2$, where two values of a many-valued function come together.

This suggests the change of variable $1-e=t^2$; then S as a function of t may be treated by successive differentiation; or, what is at bottom the same thing, let the various terms in the brace be expanded in integral powers of t, with the values at $t=0$ suitably chosen thus:

$$\sqrt{[1-e^2]}=\sqrt{[2]}\,t-\tfrac{1}{4}\sqrt{[2]}\,t^3\ldots.$$

$$e\sqrt{[1-e^4]}=2t-\tfrac{7}{2}t^3\ldots.$$

$$\sin^{-1}e=\frac{\pi}{2}-\sqrt{[2]}\,t-\frac{\sqrt{2}}{12}t^3\ldots.$$

$$\sin^{-1}e^2=\frac{\pi}{2}-2t+\tfrac{1}{6}t^3\ldots.$$

then in the expression for S, the factors t^3, to which the indeterminateness is due, may be cancelled, and the leading coefficients are then such as to show that the limit of S is S_1 as t approaches zero.

A contrasting case occurs with respect to the entire area of the ellipsoid, which is

$$2\pi a^2\sqrt{[1-e^2]}\left\{\frac{\sin^{-1}e}{e}+\sqrt{[1-e^2]}\right\},$$

the limit of which, as e approaches zero, is the area of a sphere of radius a. Here the indeterminateness occurring in $\dfrac{\sin^{-1}e}{e}$ is of integral order.

Many similar examples could be cited, for instance, that of the center of gravity or center of pressure of part of an ellipse and the corresponding part of the limiting parabola, where the limit of an indeterminate form, discovered by geometric or physical considerations, is to be analytically verified.